

ON NOETHERIAN PRIME RINGS⁽¹⁾

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A ring S with identity element is a *classical right quotient ring of a ring R* in case: (i) $S \supseteq R$; (ii) R contains nondivisors of zero, called regular elements, and each regular $d \in R$ has a two-sided inverse $d^{-1} \in S$; and (iii)

$$S = \{ab^{-1} \mid a, \text{ regular } b \in R\}.$$

Classical left quotient rings are defined symmetrically. R is *right* (resp. *left*) *quotient-simple* in case R has a classical right (resp. left) quotient ring S which is isomorphic to a complete ring D_n of $n \times n$ matrices over a (not necessarily commutative) field D . R is *quotient-simple* if R is both left and right quotient-simple.

Goldie [2] has determined that a ring R is right quotient-simple if and only if R is a prime ring satisfying the maximum conditions on complement and annihilator right ideals. In particular, any right noetherian prime ring is right quotient-simple. (See also Lesieur-Croisot [1].)

A (not necessarily commutative) integral domain K is a *right Ore domain* in case K possesses a classical right quotient field $\hat{K}$. Observe that if K is a right Ore domain, then, for each natural number n , the ring K_n of all $n \times n$ matrices over K is right quotient-simple, and $(\hat{K})_n$ is its classical right quotient ring.

A consequence of our main result (Theorem 2.3) is that the right quotient-simple rings can be determined as the class of intermediate rings of the extensions $(\hat{K})_n$ over K_n , n ranging over all natural numbers, and K ranging over all right Ore domains. Theorem 2.3 is much more precise. As a corollary we rederive a theorem of Goldie [3] on principal right ideal prime rings.

1. General quotient rings. If R is any ring, M_R (resp. ${}_R M$) will denote that M is a right (resp. left) R -module. If N is a submodule of M_R such that any nonzero submodule P of M has nonzero intersection with N , then M is an *essential extension of N* , or N is an *essential submodule of M* . Notation: $M \nabla N$ or $(M \nabla N)_R$. A right ideal I of R satisfying $(R \nabla I)_R$ is an *essential right ideal*.

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If A is a ring containing R , then A is a *right quotient ring* of R in case $(A \nabla R)_R$, where the module A_R is defined in the natural way. Any classical right quotient ring S of a ring R is a right quotient ring in this sense, so the results of this section are applicable to classical quotient rings.

1. **THEOREM.** *Let R be a semiprime ring, and let A be a right quotient ring of R . Let e be an idempotent of A such that $D = eAe$ is a field ⁽³⁾. If*

$$K = eAe \cap R \neq 0,$$

then K is a right Ore domain, and D is its right quotient field,

$$D = \{kq^{-1} | k, 0 \neq q \in K\}.$$

Proof. First note that eA is a left vector space over $D = eAe$. Thus if $0 \neq d \in D$, then $dx = 0 \implies x = 0 \forall x \in eA$. Let $[d]$ denote the R -submodule of A generated by $d \in D$. If we set $d(r, n) = dr + nd \forall (r, n) \in R \times \mathbf{Z}$ (cartesian product), $r \in R$, $n \in \mathbf{Z}$, then $[d] = \{d(r, n) | (r, n) \in R \times \mathbf{Z}\}$. Since $(A \nabla R)_R$, we have $[d] \cap R \neq 0 \forall 0 \neq d \in D$. Hence let

$$0 \neq c = d(r, n) \in [d] \cap R.$$

Since $d = de$, clearly $e(r, n) \neq 0$, so choose $(r', n') \in R \times \mathbf{Z}$ such that

$$0 \neq e(r, n)(r', n') \in R.$$

Then setting $U = eA \cap R$, and setting $a = e(r, n)(r', n')$, we see that $a \in U$ and $b = da = c(r', n') \in U$. Since $a \neq 0$, necessarily $b \neq 0$. Since $V = Ae \cap R$ is a left ideal of R , $K = eAe \cap R = V \cap U$ is a left ideal of U . Furthermore, since $K \neq 0$ by hypothesis, since $K \subseteq D$, and since $U \subseteq eA$, U is a torsion-free left K -module. Since R is semiprime, the left annihilator ideal of R is zero, so $aR \neq 0$. Since $aR \subseteq U$, it follows that $KaR \neq 0$. Since KaR is a right ideal of R , semiprimeness of R implies that $(KaR)^2 \neq 0$, hence $aRK \neq 0$. Now choose $t \in R$, $q \in K$ such that $x = atq \neq 0$. Since $x \in eA$, necessarily $y = dx \neq 0$. Since $at, bt \in U$ (U is a right ideal of R and $a, b \in U$), necessarily $x = atq$, $y = btq \in K$ (K is a left ideal of U and $q \in K$). If x^{-1} denotes the inverse of x in D , we see that $d = yx^{-1}$, and $x, y \in K$. Thus, D is the right quotient field of K , completing the proof.

REMARK. Any quotient ring A of a semiprime ring R is semiprime. Thus (see Jacobson [1, p. 65, Proposition 1]) eAe is a field if and only if eA is a minimal right ideal.

If R is any ring, and if x^r is the right annihilator in R of $x \in R$, then

$$Z_r(R) = \{x \in R | x^r \text{ is an essential right ideal of } R\}$$

is a two-sided ideal of R (R. E. Johnson [1]), called the *right singular ideal*

⁽³⁾ In our terminology, a field is not necessarily commutative.

of R . It is easy to check that $Z_r(R)$ contains no nonzero idempotents, so that $Z_r(R) = 0$ whenever R is a (von Neumann) regular ring, in particular when R is semisimple artinian.

Below we show that the vanishing of $Z_r(Q)$ is enough to insure transitivity of the relation "quotient ring of". In the proof, if x is an element in the ring Q , and if P is a subring of Q , then

$$(P: x) = \{p \in P \mid xp \in P\}$$

is a right ideal of P . Furthermore, if $(Q \nabla P)_P$, then $(P: x)$ is an essential right ideal of P , a fact which we use without proof.

2. **LEMMA.** *Let Q be a right quotient ring of R , and let R be a right quotient ring of T . (1) If I is any right ideal of Q such that $I \cap R$ is an essential right ideal of R , then I is an essential right ideal of Q ; (2) $Z_r(Q) \supseteq Z_r(R) \supseteq Z_r(T)$; (3) If $Z_r(Q) = 0$, then Q is a right quotient ring of T .*

Proof. (1) is trivial. (2) Let x' denote the right annihilator in Q of $x \in Q$. If $x \in Z_r(R)$, then $x' \cap R$ is an essential right ideal of R . Then (1) implies that $x \in Z_r(Q)$, proving (2).

(3) If $x \in Q$, $xR = 0$ implies by (1) that $x \in Z_r(Q)$. Since $Z_r(Q) = 0$, if $0 \neq x \in Q$, then $xR \neq 0$, so $xR \cap R \neq 0$. Let $s, r \in R$ be such that $s = xr \neq 0$. Now $(T: r)$ (resp. $(T: s)$) is an essential right ideal of T , and so is

$$(T: r) \cap (T: s).$$

Hence $s' \supseteq (T: r) \cap (T: s)$ would imply by (1) that $s \in Z_r(R)$. But $Z_r(R) = 0$ by (2) and $s \neq 0$, so we conclude that $s' \not\supseteq (T: r) \cap (T: s)$. Accordingly we can choose $t \in (T: r) \cap (T: s)$ such that $st \neq 0$. Then $st = x(rt) \in xT \cap T$, so $xT \cap T \neq 0$. This proves (3).

For convenience, we recall the definition of a prime ring. R is said to be prime in case any of the following three equivalent conditions are satisfied:

- (a) $I^r = 0 \forall$ right ideals I ;
- (b) $I^l = 0 \forall$ left ideals I ;
- (c) $xRy = 0 \implies x = 0$ or $y = 0 \forall x, y \in R$.

Here $I^r = \{a \in R \mid Ia = 0\}$, and $I^l = \{a \in R \mid aI = 0\}$.

If A (resp. B) is a left (resp. right) ideal of R , then $T = BA$ is defined to be the set of all finite sums of the products ba , $a \in A$, $b \in B$. It is to be observed that T is a subring of R .

3. **PROPOSITION.** *Let $R \neq 0$ be a prime ring, let A be a left ideal of R whose right annihilator A^r in R is zero, and let B be a right ideal of R whose left annihilator B^l in R is zero. Then: (1) $T = BA$ is a prime ring; (2) If, in addition, B is an essential right ideal of R , then R is a right quotient ring of T .*

Proof. (1) Let $x, y \in T$ be such that $xTy = 0$. Then $AyRx$ is an ideal of R

having square zero, so primeness of R yields $AyRxB = 0$. Then again by primeness of R , $Ay = 0$, or $xB = 0$. Since $A^r = B^l = 0$, we obtain $y = 0$ or $x = 0$, and T is therefore prime.

(2) If $0 \neq x \in R$, then $xB \neq 0$. Then $(R \nabla B)_R$ implies $xB \cap B \neq 0$. Let $b \in B$ be such that $0 \neq xb \in B$. Primeness of R implies $A^l = 0$, so $xbA \neq 0$. But $bA \subseteq T$, and $xbA \subseteq T$, so $xT \cap T \neq 0$, proving (2).

2. Quotient-simple rings. Before proving the main result (Theorem 2.3) we list some known properties of classical quotient rings.

1. LEMMA. *Let Q be a classical right quotient ring of a ring R . Then: (1) If $b_1, \dots, b_n$ are regular elements of R , there exists a regular element $c \in R$ and elements $g_i \in R$ such that $b_i^{-1} = g_i c^{-1}$, $i = 1, \dots, n$; (2) If $x_1, \dots, x_n \in Q$, there exists a regular element $c \in R$ such that $x_i c \in R$, $i = 1, \dots, n$; (3) If I is a right ideal of R , then the right ideal of Q generated by I is IQ , and $IQ = \{xc^{-1} | x \in I, \text{ regular } c \in R\}$; (4) If $d \in R$ is regular, then dR is an essential right ideal of R ; (5) If R has a classical left quotient ring, then Q is a classical left quotient ring of R .*

These results occur various places in Goldie's paper [1], but none require any of the deeper results found there. For example (1) is an easy induction [1, p. 605, Lemma 4.2], (2) is an immediate consequence of (1), and (3) follows from (2) [1, p. 605, Lemma 4.3]. (4) is [1, p. 603, Theorem 10] but a shorter argument is as follows: Let $I = dR$. Since $1 = dd^{-1} \in IQ$, then $IQ = Q$. Hence if $0 \neq k \in Q$, then (3) implies that $k = xc^{-1}$ with $x \in I$, $c \in R$. Then $0 \neq kc = x \in kR \cap I$, proving (4). (5) is obvious.

We also require the following:

2. LEMMA. *If R is right quotient-simple, then R is prime.*

Although this is not explicitly stated in Goldie's paper [2], it follows from [2, p. 213, Theorem 4.4] that R is semiprime. Let J be a right ideal of R such that $I = J^r \neq 0$. Then, if Q is the classical right quotient ring of R , $IQJ \cap R$ is a right ideal of R having square equal 0. Thus $IQJ \cap R = 0$ by semiprimeness of R . Since $(Q \nabla R)_R$, we obtain $IQJ = 0$. By simplicity (that is, primeness) of Q , we conclude that $J = 0$, and R is therefore prime.

If $Q = D_n$ is the complete ring of $n \times n$ matrices over a field D , then there exists a set $M = \{e_{ij} | i, j = 1, \dots, n\}$ of matrix units of Q , and the set of elements of Q which commute with each element of M is a field isomorphic to D . Without loss of generality we can assume that this field is D ; we call it the centralizer of M in Q . If x is any invertible element of Q , then $x^{-1}Mx$ is a set of matrix units in Q whose centralizer is $x^{-1}Dx$. We call any such set a complete set of matrix units of Q .

3. THEOREM. *Let R be a right quotient-simple ring with quotient ring $Q = D_n$, D a field. (1) Then Q contains a complete set $M = \{e_{ij} | i, j = 1, \dots, n\}$ of matrix*

units with the following property: if D is the centralizer of M in Q , then R contains a subring

$$F_n = \sum_{i,j=1}^n F e_{ij},$$

where F is a right Ore domain contained in $R \cap D$ and D is the right quotient field of F . Furthermore:

$$Q = \{ak^{-1} \mid a \in F_n, 0 \neq k \in F\}.$$

(2) If R is also left quotient-simple, then every complete set M of matrix units has the property described in (1), and each corresponding D is also the left quotient field of F . Finally,

$$Q = \{q^{-1}b \mid b \in F_n, 0 \neq q \in F\}.$$

Proof. We give a proof of (1) and (2) simultaneously by showing if $M = \{e_{ij} \mid i, j = 1, \dots, n\}$ is any complete set of matrix units of Q such that

(*) there exists a regular element $y \in R$ such that $yM \subseteq R$

then M has the property in statement (1).

Now if R is also left quotient-simple, then Q is a classical left quotient ring of R , and the right-left symmetry of (2) of 2.1 asserts that each full set M has property (*).

Next assume only that R is right quotient-simple, and let N be a complete set of matrix units of Q . Then, by 2.1, there exists a regular $y \in R$ such that $Ny \subseteq R$. Hence, $M = y^{-1}Ny$ is a complete set of matrix units of Q satisfying (*).

Accordingly let $M = \{e_{ij} \mid i, j = 1, \dots, n\}$ be any complete set of matrix units of Q satisfying (*). Then, by 2.1 there exists regular $x \in R$ such that $Mx \subseteq R$. Hence the left ideal $A = \{r \in R \mid rM \subseteq R\}$ contains the regular element $y \in R$, and the right ideal $B = \{r \in R \mid Mr \subseteq R\}$ contains the regular element $x \in R$. Furthermore, B is an essential right ideal of R by (4) of 2.1.

Since R is a prime ring by 2.2, we apply 1.3 to conclude that $T = BA$ is a prime ring and that R is a right quotient ring of T . Since $Z_r(Q) = 0$, we deduce from (3) of 1.2 that Q is a right quotient ring of T .

Next we show that $e_{11}Qe_{11} \cap T \neq 0$. Now $0 \neq ye_{11} \in R$ and

$$ye_{11}M \subseteq yM \subseteq R$$

which shows that $ye_{11} \in A$. Since $x \in B$ and since x is regular it follows that $0 \neq xye_{11} \in T = BA$, so that $T \cap Qe_{11} \neq 0$. Since Q is a right quotient ring of T , $e_{11}Q \cap T \neq 0$. Then primeness of T implies that $(e_{11}Q \cap T)c \neq 0$, where $c = xye_{11}$. If $d \in e_{11}Q \cap T$ is such that $dc \neq 0$, then $dc \in T \cap e_{11}Qe_{11}$, proving our assertion.

Since $F_1 = e_{11} Q e_{11} \cap T \neq 0$, and since $e_{11} Q e_{11}$ is a field ($\cong D$), Theorem 1.1 implies that $e_{11} Q e_{11}$ is the right quotient field of $F_1 = e_{11} Q e_{11} \cap T$. Since D is isomorphic to $De_{11} = e_{11} Q e_{11}$ under the map $\phi: d \rightarrow de_{11}$, $d \in D$, this shows that D is the right quotient field of $F = \phi^{-1} F_1$. Furthermore,

$$Fe_{ij} = e_{i1} F_1 e_{1j} \subseteq e_{i1} T e_{1j} = (e_{i1} B)(A e_{1j}) \subseteq R R \subseteq R,$$

$i, j = 1, \dots, n$. Thus, R contains the subring

$$F_n = \sum_{i,j=1}^n F e_{ij},$$

and $F \subseteq R \cap D$.

If $a = \sum_{i,j=1}^n e_{ij} d_{ij} \in Q$, $d_{ij} \in D$, $i, j = 1, \dots, n$, then by 2.1, there exists $0 \neq k \in F$ such that $d_{ij} k = q_{ij} \in F$, $i, j = 1, \dots, n$. Then $a = f k^{-1}$, where $f = \sum_{i,j=1}^n e_{ij} q_{ij} \in F_n$. This proves (1).

If R is also left quotient-simple, then Q is the classical left quotient ring of R , and Theorem 1.1 implies that $e_{11} Q e_{11}$ (resp. D) is the left quotient field of F_1 (resp. F). The computation above establishes that if $a \in Q$, then $a = q^{-1} b$, with $b \in F_n$, and $0 \neq q \in F$. This completes the proof of (2).

4. COROLLARY (A. W. GOLDIE [3]). *If R is a principal right ideal ring, and if R is prime, then $R = K_n$, where K is a right Ore domain.*

Proof. Using the notation of the theorem, we can write $B = cR$ for some $c \in R$. If $b \in Q$ is such that $bc = 0$, then $bB = 0$. But $x \in B$ is regular in R , so $x^{-1} \in Q$. Thus $bx = 0$ and $b = 0$, so c is not a right zero divisor in Q . Then, as is well known in artinian rings, $c^{-1} \in Q$.

Trivially $e_{ij} B \subseteq B$, that is, $e_{ij} cR \subseteq cR$ and $c^{-1} e_{ij} c = f_{ij} \in R$, $i, j = 1, \dots, n$. If G is the centralizer of $N = c^{-1} M c$ in Q , it follows that

$$R = \sum_{i,j=1}^n K f_{ij},$$

where $K = G \cap R$. Since Q is the classical right quotient ring of R , an easy computation shows that G is the right quotient field of K . (This fact also follows from the theorem since the theorem states that G is the right quotient field of some integral domain contained in K .)

At present (see Goldie [3]) it is unknown whether or not K has to be a principal right ideal domain. R. Bumby has shown us that the answer is "yes" if K is commutative.

3. Supplementary remarks. (A) Let R be right quotient-simple with right quotient ring $Q = D_n$, $n > 1$. If f is any idempotent of Q such that fQ is a minimal right ideal, then $f = ac^{-1}$, with a , regular $c \in R$. Thus $e = c^{-1}fc$ is idempotent and $0 \neq ce \in Qe \cap R$. Thus, $Qe \cap R \neq 0$, and since $eQ \cap R \neq 0$,

it follows from primeness of R that $(eQ \cap R)(Qe \cap R) \neq 0$, so that

$$eQe \cap R \neq 0.$$

Then, Theorem 1.1 implies that $K = eQe \cap R$ is a right Ore domain, and $\hat{K} = eQe$ is its right quotient field. Since $\hat{K} \cong D$, we obtain that $Q \cong \hat{K}_n$, where K is a right Ore domain contained in R . This illustrates the precise nature of Theorem 2.3, which states much more.

(B) Next we show that (2) of Theorem 2.3 fails without the hypothesis that R is also left quotient-simple. The example below was suggested by S. U. Chase.

Let K be a right Ore domain which is not a left Ore domain (e.g., Goldie [2, p. 219]), let x, y be nonzero elements of K such that $Kx \cap Ky = 0$, and let

$$R = \begin{pmatrix} Kx & Ky \\ Kx & Ky \end{pmatrix}.$$

(R is the ring of all 2×2 matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with $a, c \in Kx$, and $b, d \in Ky$.) Since K is right Ore, if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is an arbitrary element of R , there exists $0 \neq q \in K$ such that $aq, bq, cq, dq \in K$, and then

$$B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} qx & 0 \\ 0 & qy \end{pmatrix} \in R.$$

Thus, $A = BC^{-1}$, with

$$B, C = \begin{pmatrix} qx & 0 \\ 0 & qy \end{pmatrix} \in R.$$

Hence, $\hat{K}_2$ is the classical right quotient ring of R , that is, R is right quotient-simple.

As in Theorem 2.3, we identify $\hat{K}$ with the subring of $\hat{K}_2$ consisting of all scalar matrices $\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$ with $k \in \hat{K}$. Now assume for the moment that R contains a subring F_2 where F is an integral domain $\subseteq \hat{K}$. The contradiction is immediately evident (even without assuming that $\hat{F} = \hat{K}$), since the form of R ,

$$R = \begin{pmatrix} Kx & Ky \\ Kx & Ky \end{pmatrix},$$

where $Kx \cap Ky = 0$, precludes the possibility of its containing a nonzero scalar matrix $\begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix}$ with $0 \neq d \in \hat{K}$.

(C) Theorem 2.3 implies that a right quotient-simple ring R which is not an integral domain contains nonzero nilpotent elements. However such a ring R need not contain nontrivial idempotents even if R contains an identity. Perhaps the simplest example is as follows: Let $S = Q_2$ be the ring of all 2×2 matrices over the rational number field Q , and let R be the subring consisting of all matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where b, c are even integers, and a, d are integers which are either both even or both odd. Then $R = (2\mathbb{Z})_2 + \mathbb{Z}$ is not an integral domain. However R is quotient-simple with quotient ring S , with an identity 1, and R does not contain idempotents $\neq 0, 1$.

(D) Now let R be a ring having a classical right quotient ring S which is semisimple artinian. If T is a simple component of S , then $T = eS$, where e is a central idempotent of S , and it can be shown that T is the classical right quotient-ring of $eR \cap R$. Thus, by the theorem, R contains a direct sum of finitely many full rings of matrices K_i over various right Ore domains K , and S is the direct sum of the corresponding simple components $(\hat{K})_i$.

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